Chapter 22

Magnetic Forces and Magnetic Fields

The Cross Product of Two Vectors

When you multiply any two vectors $\vec{A}$ and $\vec{B}$ via the "cross product", the result is a third vector $\vec{C}$ such that $\vec{C} = \vec{A} \times \vec{B}$. What are the magnitude and direction of the cross product $\vec{C}$?

(a) The magnitude of $\vec{C}$ is given by

$$|\vec{C}| = |\vec{A}| |\vec{B}| \sin(\theta)$$

where $\theta$ is the smallest angle between the two vectors $\vec{A}$ and $\vec{B}$.

(b) The direction of $\vec{C}$ is

1. perpendicular to both vectors $\vec{A}$ and $\vec{B}$. This narrows down the choices for the direction of vector $\vec{C}$ to two possible directions.
2. Use the right-hand-rule to choose between these two possible choices for the direction of $\vec{C}$. Simply point the 4 fingers of your right hand in the direction of the first vector in the cross product, namely $\vec{A}$, and aim the palm of your right hand in the direction of the second vector in the cross product, namely $\vec{B}$. The thumb of your right hand then points in the direction of the cross product $\vec{C}$.

22.2 The Magnetic Field $\vec{B}$

When a point charge $q$ moves with velocity $\vec{v}$ in an external magnetic field $\vec{B}_{ext}$, the magnetic field exerts a force $\vec{F}$ on the point charge given by

$$\vec{F} = q \vec{v} \times \vec{B}_{ext}$$

The "SI" unit of magnetic field is the Tesla.
22.5 Magnetic Force on a Current-Carrying Conductor

A current-carrying wire also experiences a force when it is placed in an external magnetic field $\vec{B}_{ext}$. The force on a wire of length $\ell$, current $I$, in an external magnetic field $\vec{B}_{ext}$ is given by

$$\vec{F} = \ell I \times \vec{B}_{ext}$$

22.6 Torque on a Current Loop in a Uniform Magnetic Field

Consider a loop with an area $A$ and carrying a current $I$. The area vector $\vec{A}$ is a vector whose magnitude is the area of the plane of the loop, and whose direction is perpendicular to the plane of the loop. Determine the direction of the area vector $\vec{A}$ using the right-hand-rule. Curl 4 fingers of right hand in the direction of the current around the loop, then the thumb points in the direction of the area vector $\vec{A}$. 
The *magnetic dipole moment* $\bar{\mu}$ of the loop is defined as

$$\bar{\mu}_{\text{one loop}} = I \bar{A}$$

The *torque* $\tau$ exerted on a current-carrying loop placed in an external magnetic field $\bar{B}_{\text{ext}}$ is

$$\tau = \bar{\mu}_{\text{one loop}} \times \bar{B}_{\text{ext}}$$

If it is a coil with $N$ turns, then $\bar{\mu}_{\text{coil}} = N \bar{\mu}_{\text{one loop}}$ and so

$$\Rightarrow \quad \bar{\tau}_{\text{coil}} = N \bar{\mu}_{\text{one loop}} \times \bar{B}_{\text{ext}}$$

The coil will be rotated by $\bar{B}_{\text{ext}}$ so that ultimately $\bar{\mu}_{\text{coil}}$ is parallel to $\bar{B}_{\text{ext}}$. This corresponds to the configuration of *minimum energy*. This is so because the potential energy $U_{\text{magnetic}}$ stored in the system consisting of the current-carrying coil in the external magnetic field equals

$$U_{\text{magnetic}} = -\bar{\mu}_{\text{coil}} \cdot \bar{B}_{\text{ext}}$$

(a) lowest energy $\Rightarrow U_{\text{minimum}} = -\mu_{\text{coil}} B_{\text{ext}}$ when $\bar{\mu}_{\text{coil}}$ is parallel to $\bar{B}_{\text{ext}}$.

$$U_{\text{mag.}} = -\mu_{\text{coil}} B_{\text{ext}} \cos(\theta)$$

$$U_{\text{mag.}} = -\mu_{\text{coil}} B_{\text{ext}}$$

$\bar{\mu}_{\text{coil}} \rightarrow \bar{B}_{\text{ext}} \quad \theta = 0^\circ$

*Lowest energy configuration*
(b) highest energy ⇒ $U_{\text{maximum}} = +\mu_{\text{coil}} B_{\text{ext}}$ when $\mu_{\text{coil}}$ is antiparallel to $\vec{B}_{\text{ext}}$.

\[ \mu_{\text{coil}} \quad \theta = 180^\circ \quad \rightarrow \quad \vec{B}_{\text{ext}} \]

$U_{\text{mag}} = -\mu_{\text{coil}} B_{\text{ext}} \cos(180^\circ)$

$U_{\text{mag}} = +\mu_{\text{coil}} B_{\text{ext}}$

This corresponds to the maximum energy configuration.

22.3 **Motion of a charge particle in a uniform magnetic field**

Consider a charged particle of mass $m$ moving with velocity $\vec{v}$ in an external magnetic field $\vec{B}_{\text{ext}}$ that is directed into the page as indicated by the x's. Suppose $\vec{v}$ and $\vec{B}_{\text{ext}}$ are perpendicular to each other. Thus

\[ \sum F_c = ma_c \]

$F_{\text{magnetic}} = \frac{m v^2}{r}$

$q v B_{\text{ext}} \sin(90^\circ) = \frac{m v^2}{r}$

$r = \frac{m v}{q B_{\text{ext}}}$.

The magnetic force on the charged particle is always perpendicular to the velocity vector so the particle moves in a circular orbit of radius $r$ equal to:

\[
\boxed{r = \frac{m v}{q B_{\text{ext}}}}
\]
The angular speed $\omega = \frac{v}{r}$ becomes

$$r = \frac{m v}{q B_{\text{ext}}}$$

$$\omega = \frac{q B_{\text{ext}}}{m}$$

This is known as the *cyclotron frequency* $\omega$. Cyclotrons are used to accelerate charged particles to very high speeds. The *period* of the motion $T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$ is

$$T = \frac{2\pi m}{q B_{\text{ext}}}$$

Note that both $\omega$ and $T$ are independent of the radius $r$ and the speed $v$. This is crucial in the operation of a cyclotron.

Van Allen Radiation Belts: Charged particles (mostly emitted by the Sun) collide with atoms in the atmosphere and cause the atoms to emit visible light. Aurora Borealis are the northern lights. Aurora Australis are the southern lights.
22.4 Applications Involving Charged Particles Moving in an External Magnetic Field

A. Velocity Selector
See figure 22.1.
$B_{ext}$ is directed into the page. The positively charged particles will move *undeflected* by the external electric and magnetic fields provided they move with a speed $v$ that satisfies:

$$v = \frac{E_{ext}}{B_{ext}}$$

This is so because:

$$\sum F_y = 0 \text{ if undeflected}$$

$$F_{magnetic} - F_{electric} = 0$$

$$q v B_{ext} - q E_{ext} = 0$$

$$v = \frac{E_{ext}}{B_{ext}}.$$

B. Mass Spectrometer
It separates ions according to their mass-to-charge ratio. A beam of ions first passes through a velocity selector with crossed $E$ and $B$ fields, then on to a uniform magnetic field $B_o$ region:

$$\sum F_c = m a_c$$

$$q v B_o = m \frac{v^2}{r}$$

$$m = \frac{B_o r}{v} \quad \text{but} \quad v = \frac{E}{B} \quad \text{so}$$

$$m = \frac{B_o r}{E/B} = \frac{B B_o r}{E} \quad (7)$$
Determine the m/q ratio by measuring the radius of curvature r and from knowledge of the magnitudes of the externally applied fields B, B₀, and E. The result is

\[
\frac{m}{q} = \frac{B B₀ r}{E}
\]

22.9 **Ampere’s Law**

The line integral of \( \vec{B} \cdot d\vec{s} \) around any closed path equals \( \mu₀ I_{enc} \) where I_{enc} is the total (net) current passing through any surface bounded by the closed path. Mathematically we can write this as

\[
\oint \vec{B} \cdot d\vec{s} = \mu₀ I_{enc}
\]

Amperian Loop

Here one imagines an Amperian loop, much like we did when we imagined Gaussian surfaces. \( \vec{B} \) is the magnetic field at the Amperian loop, \( d\vec{s} \) is taken along the perimeter of the Amperian loop, and \( I_{enc} \) is the net current passing through the Amperian loop.

Ampere’s law holds true in magnetostatics (time independent magnetic fields produced by currents that are constants over time).
22.8 Magnetic Force Between Two Parallel Conducting Wires

Consider two very long, straight, parallel wires separated by a distance "a" and carrying currents $I_1$ and $I_2$ in the same direction. What is the force exerted on one wire by the magnetic field generated by the other wire?

\[ \vec{F}_{on1} = l_1 \vec{I}_1 \times \vec{B}_{at1} \]

where \( \vec{B}_{at1} = \frac{\mu_0 I_2}{2\pi a} \)

hence \( \vec{F}_{on1} = I_1 [I_1 \uparrow] \times [\frac{\mu_0 I_2}{2\pi a} \downarrow] \)

\[ \frac{\vec{F}_{on1}}{l_1} = \frac{\mu_0 I_1 I_2}{2\pi a} \]

Thus

\[ \frac{\vec{F}_{on1}}{l_1} = \frac{\mu_0 I_1 I_2}{2\pi a} \quad (\text{right or attractive}) \]

If currents are in the same direction $\Rightarrow$ wires attract.
If currents are in opposite directions $\Rightarrow$ wires repel.
22.10 The Magnetic Field of a Solenoid

Consider an infinitely long solenoid. One for which the turns are closely spaced and whose length is much greater than the radius of the turns.

Suppose the solenoid has $n$ turns per unit length, and that the current in the solenoid is $I$. Then the magnetic field generated by the solenoid is uniform and axial inside and zero outside the solenoid. The magnetic field inside the solenoid has a magnitude equal to $B = \mu_0 n I$ and a direction along the axis of the solenoid as determined by the right-hand-rule. Curl 4 fingers of right hand in the direction of the current and the thumb points in the direction of the magnetic field. See derivation below.

\[ B \cdot d\vec{s} + \int_{\theta=0}^{\theta=90} B \cdot d\vec{s} + \int_{\phi=0}^{\phi=90} B \cdot d\vec{s} + \int_{\phi=90}^{\phi=180} B \cdot d\vec{s} = N_0 (n I L) \]

\[ B \cdot L = N_0 n I \]

Axial and uniform inside, zero outside.

\[ B \approx 0 \text{ (far away)} \]