Galilean Relativity

\[ x = x' + ut \]

\[ \frac{dx}{dt} = \frac{dx'}{dt} + u \]

- \( v = v_0 + u \) if \( t = t' \) \( \Rightarrow \) time is absolute

\[ \frac{dv}{dt} = \frac{dv'}{dt'} = \frac{dv}{dt'} \] or \( a = 0 \)

\[ F = ma = ma' = \mu(x' + u') \] if \( m = m' \)

\[ m\dddot{x}_1 + \mu\dddot{x}_1 = m\dddot{x}_1' + \mu\dddot{x}_1' \]

\[ m(\dddot{x}_2 + u) + \mu(\dddot{x}_2 + u) \]

\[ = m(\dddot{x}_2 + u) \]

\[ \mu(\dddot{x}_2 + u) \]

\[ m\dddot{x}_2 + \mu\dddot{x}_2 + \mu a' \]

\[ = m\dddot{x}_2' + \mu\dddot{x}_2' + \mu a' \]

\[ m\dddot{x}_2 + \mu\dddot{x}_2 = \text{Constant} \]

\[ m\dddot{x}_2 + \mu\dddot{x}_2 = 0 \]

\[ F_{m1} + F_{M1} = 0 \]

\[ F_{m1} = -F_{M1} \]

3rd Law is Satisfied.
So, all laws of mechanics are Galilean invariant. Applied to light, we have:

\[ C = C' + u \text{ or } u = C - C' \]

\[ \Rightarrow \text{if the speed of light could be measured in two inertial frames of reference, as } C \text{ and } C', \text{respectively, then the relative speed between the two frames measured to be } u. \]

**Michelson–Morley Experiment**

\[ t_1 = \frac{u}{c + u}, \quad t_2 = \frac{u}{c - u} \]

\[ T_1 = t_1 + \frac{1}{u} \frac{1}{c + u} \]

\[ T_2 = t_2 - \frac{1}{u} \frac{1}{c - u} \]

\[ T_1 = \frac{L}{1 - \frac{v^2}{c^2}} - \frac{2v}{c} \left( 1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}} = \frac{L}{c} \left( 1 + \frac{v^2}{c^2} \right)^{-\frac{1}{2}} \]

\[ T_2 = \frac{2L}{c} \left( 1 - \frac{v^2}{c^2} \right)^{1/2} = \frac{2L}{c} \left( 1 + \frac{v^2}{c^2} \right) \]

\[ \delta T = T_1 - T_2 = \frac{2L}{c} \left[ \sqrt{1 + \frac{v^2}{c^2}} - 1 - \frac{v^2}{2c^2} \right] = \frac{2Lv^2}{2c^3} \]

\[ \delta T' = \frac{Ll^2}{c^3} \]

\[ \delta T = 2 \delta T' = \frac{2Lv^2}{c^3} \]

\[ \cos \theta = \frac{vL}{c^2} \]

\[ \Delta \theta = \frac{\Delta L}{L} \]

\[ \Delta L = \frac{2Lv^2}{c^2} \]

\[ \Delta L = \frac{2(11m)(3 \times 10^8)^2}{598 \times 10^{-9} (3 \times 10^8)^2} = \frac{22}{598} = 0.0367 \text{ fringes} \]
Lorentz Transformation is based on the two postulates:
1. All laws of physics are the same in all inertial frames of reference, and
2. The speed of light is the same in all inertial frames of reference.

\[
\begin{align*}
  x &= Ax' + Bt' \\
  t &= D x' + E t' \\
  y &= y' \quad z = z'
\end{align*}
\]

Motion of origin $S$ (x=0) as viewed by $S'$ is $x = ct$. Therefore:
\[
ct = bt' \quad \text{and} \quad t = Et'. \quad t = \frac{E}{c} \\
ct = \frac{Bt'}{E} \quad B = \frac{ct}{E}
\]

So far:
\[
\begin{align*}
  x &= Ax' + ct \frac{A}{c} t' = A (x' + ct') \\
  t &= Dx' + \frac{Ut}{c} = Dx' + At'
\end{align*}
\]

Motion of light as viewed by $S$ and $S'$ are $x = ct$, and $x = ct'$, respectively.
\[
ct = A (ct' + Ut) = A ct' + Au t' \\
ct = Dc t' + At' \\
oc = Au t' - Dc t' \quad \text{or} \quad D = \frac{Au}{c^2}
\]

Hence so far:
\[
\begin{align*}
  x &= A (x' + ut) \\
  t &= A (t' + \frac{Ut}{c^2})
\end{align*}
\]

The inverse transformation is:
\[
\begin{align*}
  x' &= A (x - ut) \\
  t' &= A (t - \frac{Ut}{c^2}) \\
  y' &= y \quad z' = z
\end{align*}
\]
Transformation followed by its inverse must yield identity:

\[ x = A \left[ A (x - ut) + u A \left( t - \frac{u^2}{c^2} x \right) \right] \]

\[ = A \left[ A x A u^2 + A u t - \frac{u^2}{c^2} x \right] = A^2 \left[ 1 - \frac{u^2}{c^2} \right] x \]

\[ A^2 = \frac{1}{1 - \frac{u^2}{c^2}}, \quad A = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma \]

Finally:

\[ x' = x \left( x' + ut' \right), \quad y = y', \quad t' = \frac{t}{\gamma}, \quad s = \left( 1 - \frac{u^2}{c^2} \right)^{-1/2} \]

Some Consequences \#1.

a. Length contraction.

\[ x_b = \gamma \left[ x_b' + u t_b' \right] \]

\[ x_A = \gamma \left[ x_A' + u t_A' \right] \]

\[ x_b' = s \left( x_b' + u t_b' \right) \]

\[ x_A' = s \left( x_A' + u t_A' \right) \]

\[ u_b - \gamma x_A' = \gamma \left[ x_b' - x_A' + u (t_b' - t_A') \right] \]

\[ = \gamma \left[ x_b' + u (t_b' - t_A') \right] \]

\[ t_b' = \gamma \left[ t_b' + \frac{u^2}{c^2} x_b' \right] \]

\[ t_A' = \gamma \left[ t_A' + \frac{u^2}{c^2} x_A' \right] \]

\[ t_b' - t_A' = 0 \Rightarrow t_b' - t_A' = -\frac{u}{c^2} (x_b' - x_A') \]

\[ s \]

Each observer finds the length of the moving rod shortened.
b. **Time Dilation**

\[
\begin{align*}
    t &= \gamma (t' + \frac{u}{c^2} x) \\
    t' &= \gamma (t' - \frac{u}{c^2} x') \\
    t'' &= \gamma (t'' - \frac{u}{c^2} x'') \\
\end{align*}
\]

\[
\Delta t' = t' - t'' = \gamma \left[ t' - t'' + \frac{u}{c^2} (x' - x'') \right] = \gamma \Delta t
\]

\[
\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{u^2}{c^2}}}
\]

Suppose observer O measures the time interval between two events. Event A at \( x_A, t_A \) and event B at \( x_B, t_B \). The time interval \( \Delta t' \) between the two events as measured by O, observer in \( \Delta t = t_B - t_A \) as measured by O, observer. The time interval would be different from Eq.(i) due to a delaying time required for light to travel the distance between the two events as measured by O, observer. [Time dilation is purely inconvenient due to motion only].

**Velocity Transformation**

Let \( u \) and \( v \) be the velocities of an object as measured by observers O and O', respectively. By L.T., we have:

\[
\begin{align*}
    dx &= \gamma \left[ dx' + u dt' \right] \\
    dt &= \gamma \left[ dt' + \frac{u}{c^2} dx' \right] \\
\end{align*}
\]

\[
\frac{dx}{dt} = \frac{dt'}{(1 + \frac{u}{c^2} \frac{dv}{dt})} = \frac{v + u}{1 + \frac{u^2}{c^2}}
\]

\[
\frac{dy}{dt} = \frac{dy'}{dt'} = \frac{dy'}{y'} dt' \left( 1 + \frac{v}{c^2} \frac{dy'}{dt'} \right) = \frac{V_y}{\gamma \left( 1 + \frac{u^2}{c^2} \right)}
\]

Similarly for the component \( V_z \). Hence:

\[
\begin{align*}
    V_x &= \frac{V_{x'} + u}{1 + u V_{x'}/c^2} \\
    V_y &= \frac{\sqrt{1 - u^2/c^2} V_{y'}}{1 + u V_{y'}/c^2} \\
    V_z &= \frac{\sqrt{1 - u^2/c^2} V_{z'}}{1 + u V_{z'}/c^2}
\end{align*}
\]
In particular, if
(a) \( V_0 = C \), \( \beta = \frac{V_0}{C} = \frac{U + C}{1 + \frac{VU}{C}} = C \)
(b) \( U = C \), \( \beta = \frac{V_0}{1 + \frac{VU}{C}} = C \)
(c) \( V_\infty = U = C, \) \( V_\infty = \frac{2C}{1 + \frac{VU}{C}} = C \)

as expected by the second postulate of Special Relativity.

Momentum \( \rho = \frac{mv}{\sqrt{1 - \frac{v^2}{C^2}}} \) is the momentum of a body having a rest mass \( m \) and moving with a velocity \( v \).

Naturally, if \( \frac{VU}{C} \ll 1 \), then \( \rho \) approaches the Non-Relativistic Limit \( \rho \approx mv \).

Force
\[ F = \frac{d\rho}{dt}, \quad \rho = \frac{mv}{\sqrt{1 - \frac{v^2}{C^2}}} \]

This formula has the limit \( F = \frac{d\rho}{dt}, \rho = mv \) for non-relativistic cases.

As an application, consider the case of motion of a particle under constant force. So, let \( F = f = \text{constant} \), then integrating,

\[ \frac{d\rho}{dt} = f \implies d\rho = f dt \quad \text{or} \quad \rho = ft + A, \]

where \( A \) is the initial momentum \( \rho_0 \).

\[ \frac{mv}{\sqrt{1 - \frac{v^2}{C^2}}} = ft + A, \]

Solve for \( v \) to obtain:

\[ \frac{v^2}{1 - \frac{v^2}{C^2}} = \left( \frac{\rho_0}{m} + \frac{At}{m} \right)^2, \quad v^2 = \left( \frac{\rho_0}{C^2} - \frac{v^2}{C^2} \right)^2, \quad V_\infty^2 \left[ \frac{C^2 + (\ldots)^2}{C^2} \right] = (\ldots)^2, \quad V_\infty^2 \frac{(\ldots)^2}{C^2(\ldots)^2}, \quad V = \frac{(\ldots)}{\sqrt{1 - \frac{v^2}{C^2}}}, \quad (\ldots) = \frac{\rho_0}{m} + \frac{At}{m}. \]
\[
\frac{dx}{dt} = \frac{\left(\rho \mu + tf/\mu\right)}{\sqrt{1 + \frac{1}{c^2} \left(\rho \mu + tf/\mu\right)^2}} \\
\chi = \int \frac{\rho \mu + tf/\mu}{\sqrt{1 + \frac{1}{c^2} \left(\rho \mu + tf/\mu\right)^2}} dt + A
\]

where \( A \) is a constant of integration.

\[
\cos \theta = \frac{f}{c^2} dt, \quad d\tau = \frac{mc}{c^2} \cos \theta \, d\tau
\]

\[
\chi = \int \frac{c \sin \theta \, f \sin \theta \, d\tau}{\sqrt{1 + \sin^2 \theta}} + A = A + \frac{mc^2}{f} \int \sin \theta \, d\tau
\]

\[
\chi = A + \frac{mc^2}{f} \sqrt{1 + \frac{1}{c^2} \left(\rho \mu + tf/\mu\right)^2}
\]

At \( t = 0 \), \( \chi(0) = A + \frac{mc^2}{f} \sqrt{1 + \rho^2/mc^2} \Rightarrow 0
\]

\[
A = -\frac{mc^2}{f} \sqrt{1 + \frac{\rho^2}{mc^2}}
\]

If initial momentum was also zero, \( \rho = 0 \)

\[
\chi(t) = -\frac{mc^2}{f} + \frac{mc^2}{f} \sqrt{1 + f^2t^2/mc^2}
\]

let \( f = mg \)  \( \Rightarrow \) \( \chi(t) = -\frac{mc^2}{f} + \frac{mc^2}{f} \left[1 + \frac{g^2t^2}{2mc^2} \right]
\]

\[
= \frac{mc^2}{f} - \frac{g^2t^2}{2mc^2} = \frac{t^2}{2m} - \frac{m^2g^2}{2m} \Rightarrow \chi(t) = \frac{1}{2} g t^2 + \ldots
\]

Work-Energy Principle
\[ W = \int F \cdot dl = \Delta K \]
\[ = \int \frac{d}{dt} p \cdot dl = \int dp \cdot \frac{dl}{dt} = \int dp \cdot v, \quad v = \frac{dl}{dt} \]
\[ = \int v \cdot d\left( \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \int v \cdot \left\{ \frac{mvdv}{\sqrt{1 - \frac{v^2}{c^2}}} - m \frac{\sqrt{1 - \frac{v^2}{c^2}}}{2} \left( \frac{-2 \frac{v^2}{c^2}}{2} \right) dv \right\} \]
\[ = \int v \cdot \left\{ \frac{mdv}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{(1 - \frac{v^2}{c^2})^{3/2}}{(1 - \frac{v^2}{c^2})^{3/2}} \right\} \]
\[ = \int \frac{mvdv}{(1 - \frac{v^2}{c^2})^{3/2}} = \int d\left( \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \]
\[ = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + A, \quad \text{where} \quad A \text{ is a constant} \]
\[ A + \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = K - K_0 \]
\[ \text{at } t=0, \quad K = K_0 = 0 \quad \text{and} \quad A = -mc^2 \]
\[ K = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2 = mc^2 \left[ (1 - \frac{v^2}{c^2})^{1/2} - 1 \right] \]
\[ \approx mc^2 \left[ 1 + \frac{v^2}{2c^2} - 1 \right] = \frac{1}{2} mv^2 \quad \text{as expected for } \frac{v}{c} \ll 1.
The quantity \( E = \frac{mc^2}{\sqrt{1-v^2/c^2}} \) is the energy of a moving particle and the body at rest has a rest energy in the amount \( E = mc^2 \).

The kinetic energy is then the difference of the energy of a particle in motion and the rest energy of the particle.