Chapter 7

Potential Energy and Energy Conservation

7.1 Gravitational Potential Energy

We will write the expression for the gravitational potential energy $U_{\text{grav}}$ in either of two ways:

(A) Object of mass $m$ that is near the Earth's surface:

An object of mass $m$ at a height $y$ from some reference level has a gravitational potential energy $U_{\text{grav}}$ given by

$$U_{\text{grav}} = mg \, y \quad (y \ll R_E)$$

Comments:
1. Here $U_{\text{grav}} = 0$ at $y = 0$ and increases as $y$ increases.
2. a good approximation when $y \ll R_E$ where $R_E$ is the radius of the Earth.
(B) Object of mass \( m \) that is \textit{far} from the Earth’s surface:

\[
U_{grav} = -\frac{GM_E m}{r}
\]

where

\[
G = \text{Universal Gravitational Constant} = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}
\]

\[
M_E = \text{Mass of the Earth} = 5.98 \times 10^{24} \text{ kg}
\]

\[
r = \text{distance from the (center of mass) of object to the center of the Earth.}
\]

Comments:
1. Here \( U_{grav} = 0 \) at \( r = \infty \) and increases as one moves \textit{out} to \( r = \infty \).
2. Good all the time! Not just an approximation as in other case above.

**Question:** What is the work done by the gravitational force as an object changes position? The answer is simply given by

\[
W_{gravity} = -\Delta (U_{grav})
\]
where

1. \( U_{\text{grav}} = mg y \) (when "m" is close to the surface of the Earth).

2. \( U_{\text{grav}} = - \frac{GM_E m}{r} \) (when "m" is far from the surface of the Earth).

7.2 **Elastic Potential Energy**

A spring that is stretched (or compressed) by a small distance \( x \) from its *unstretched* position exerts a force \( F_s \) given by *Hooke's law*:

\[
F_{\text{spring}} = -k x
\]

where

\( x \) = the distance the spring has been stretched or compressed from its natural length

\( k \) = the *spring constant* which is a measure of the stiffness of the spring
The minus sign in Hooke’s law $F_s = -kx$ indicates that the force is directed toward the unstretched position of the spring.

When a spring is stretched (or compressed) by a small distance $x$, the spring has energy (elastic potential energy) stored in it. This potential energy $U_{spring}$ is given by

$$U_{spring} = \frac{1}{2}kx^2$$

7.14 The graph of elastic potential energy for an ideal spring is a parabola: $U_{el} = \frac{1}{2}kx^2$, where $x$ is the extension or compression of the spring. Elastic potential energy $U_{el}$ is never negative.

The Isolated System

Define mechanical energy $E$ as the sum of kinetic and potential energy:

$$E = KE + U_{grav} + U_{spring}$$

$$E = \frac{1}{2}mv^2 + mg\gamma + \frac{1}{2}kx^2$$
7.3 Conservative and Non-Conservative Forces

One may define a \textit{conservative} force by saying that the \textit{work} done by a conservative force on a particle moving between two points is \textit{independent} of the path taken by the particle.

For a \textit{non-conservative} force, the work done by it \textit{depends} on the actual path taken by the particle in moving between the two points.

Suppose an object of mass \( m \) moves from some initial position to some final position. See sketch below for relevant parameters:
Then the *Work-Energy Theorem* can be formulated in two ways.

(i) The work $\sum W_{\text{All forces}}$ done on the particle by the resultant force $\sum \vec{F}$ acting on it is equal to the change in kinetic energy $\Delta(KE)$ of the particle. That is

$$\sum W_{\text{All forces}} = \Delta(KE)$$

We discussed this in chapter 6.

(ii) As discussed in lecture, one can also formulate the Work-Energy theorem as follows:

$$E_i + \sum W_{\text{all forces except gravity and spring}} = E_f$$

**Note:**

1. Mechanical Energy $E$ is defined as the sum of the kinetic and the potential energies. That is,

$$E = K.E. + U_{\text{grav}} + U_{\text{spring}}$$
2. If $\sum_{\text{all forces except gravity and spring}} W = 0$, then "mechanical energy" is conserved. This is so because

$$E_i + \sum_{\text{all forces except gravity and spring}} W = E_f$$

$$E_i + 0 = E_f$$

$$E_i = E_f$$

Conservation of mechanical energy for an "isolated" system implies no energy transfer across boundaries.

Comment:
1. The gravitational force and the elastic (or spring) force are the only conservative forces that we'll learn about in physics 185. In physics 280 we will add another force to this list of conservative forces, namely the electromagnetic force.

7.4 **Conservative Forces and Potential Energy**

The hallmark of a conservative force $F$ is that we can identify a potential energy function $U$ with a conservative force. {Magnetic forces are an exception to this rule, but let us not worry about them just yet}. In cartesian
coordinates, the conservative force $\vec{F}$ is equal to the negative of the gradient of the potential energy function $U$. You will learn about the gradient operator in more advanced calculus courses. More generally,

$$\vec{F} = \left( -\frac{dU}{dx} \right)\hat{i} + \left( -\frac{dU}{dy} \right)\hat{j} + \left( -\frac{dU}{dz} \right)\hat{k}$$

Let's analyze further the gravitational force and the elastic force.

**A. The Relationship Between Gravity and Gravitational Potential Energy**

1. $U_{grav} = mgy$
   
   then $F_{grav} = -\frac{dU_{grav}}{dy}$
   
   $$= -\frac{d}{dy}(mgy)$$
   
   $$F_{grav} = -mg$$

   the negative sign indicates that the gravitational force acts downward.

2. $U_{grav} = -\frac{GM_E m}{r}$
then \[ F_{\text{grav}} = -\frac{dU_{\text{grav}}}{dr} = -\frac{d}{dr}\left(-\frac{GM_Em}{r}\right) \]

\[ F_{\text{grav}} = -\frac{GM_Em}{r^2} \]

the negative sign indicates that the gravitational force acts radially inward toward the center of the Earth (attractive).

**B. The Relationship Between the Elastic Force and Elastic Potential Energy**

Since \( U_{\text{spring}} = \frac{1}{2}k x^2 \) then

\[ F_{\text{spring}} = -\frac{dU_{\text{spring}}}{dx} = -\frac{d}{dx}\left( \frac{1}{2}k x^2 \right) \]

\[ F_{\text{spring}} = -k x \]

the negative sign indicates that the elastic force acts toward the "equilibrium" position (where \( x = 0 \)). Furthermore, consider the following:
• If \( x > 0 \), then \( \vec{F} \) is negative (to the left).
• If \( x < 0 \), then \( \vec{F} \) is positive (to the right).

Either way, the elastic (spring) force is directed toward the equilibrium position of the spring!

C. There is a **geometrical interpretation** of the relationship between a conservative force \( F \) and its potential energy function \( U \). Well, if you know the plot of potential energy \( U \) versus position \( x \), then note that

\[
F = -\frac{dU}{dx}
\]

means that the force \( F \) equals the negative of the slope of the tangent line to the curve!!! That is, consider

7.24 The maxima and minima of a potential-energy function \( U(x) \) correspond to points where \( F = 0 \).
(i) \( \vec{F} = 0 \) at points \( x_1, x_2, x_3, \) and \( x_4. \)

(ii) \( \vec{F} \) is positive in the regions \( 0 < x < x_1, x_2 < x < x_3, \) and \( x > x_4. \)

(iii) \( \vec{F} \) is negative in the regions \( x_1 < x < x_2, \) and \( x_3 < x < x_4. \)

D. Equilibrium of a System

**Stable Equilibrium:**
Configurations of *stable equilibrium* correspond to those for which the potential energy function \( U(x) \) is a minimum. Consider the elastic potential energy function of a spring that obeys Hooke’s law: \( U_{spring} = \frac{1}{2} k x^2 \)

![Graph showing the potential energy function and its derivative]

\( F_{spring} = -\frac{dU_{spring}}{dx} \) says that the elastic force is the negative of the slope of the tangent line to the potential energy curve.

- if \( x > 0 \), then \( \frac{dU_{spring}}{dx} > 0 \) so that \( F_{spring} < 0 \) (left)
- if \( x < 0 \), then \( \frac{dU_{spring}}{dx} < 0 \) so that \( F_{spring} > 0 \) (right)
• $x = 0$, then $\frac{dU_{spring}}{dx} = 0$ so that $F_{spring} = 0$ (this corresponds to \textit{stable equilibrium} when the potential energy function is a \textit{minimum} which occurs when $x = 0$ for a spring).

\textbf{Unstable Equilibrium:}
Configurations of \textit{unstable equilibrium} correspond to those for which the potential energy function $U(x)$ is a \textit{maximum}. Consider some potential energy function of a spring that looks like the following:

- if $x > 0$, then $\frac{dU}{dx} < 0$ so that $F > 0$ (right)
- if $x < 0$, then $\frac{dU}{dx} > 0$ so that $F < 0$ (left)
- $x = 0$, then $\frac{dU}{dx} = 0$ so that $F = 0$ (this corresponds to \textit{unstable equilibrium} since the potential energy function $U(x)$ is a \textit{maximum}.
